

# On $p$ -adic quaternionic Eisenstein series

Toshiyuki Kikuta and Shoyu Nagaoka

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## Abstract

We show that certain  $p$ -adic Eisenstein series for quaternionic modular groups of degree 2 become “real” modular forms of level  $p$  in almost all cases. To prove this, we introduce a  $U(p)$  type operator. We also show that there exists a  $p$ -adic Eisenstein series of the above type that has transcendental coefficients. Former examples of  $p$ -adic Eisenstein series for Siegel and Hermitian modular groups are both rational (i.e., algebraic).

## 1 Introduction

Serre [12] first developed the theory of  $p$ -adic Eisenstein series and there have subsequently been many results in the field of  $p$ -adic modular forms. Several researchers have attempted to generalize the theory to modular forms with several variables. For example, we showed that a  $p$ -adic limit of a Siegel Eisenstein series becomes a “real” Siegel modular form (cf. [4]). The same result has also been proved for Hermitian modular forms (e.g., [11]).

In the present paper, we study  $p$ -adic limits of quaternionic Eisenstein series. This study has two principal aims. The first is to show that these  $p$ -adic limits become “real” modular forms of level  $p$  for higher  $p$ -adical weights (Theorem 3.1). To prove this, we introduce a  $U(p)$  type Hecke operator and study its properties; this is a similar method to that used by Böcherer for Siegel modular forms [2]. The second aim is to show that a strange phenomenon occurs for low  $p$ -adical weights; namely, there exists a transcendental  $p$ -adic Eisenstein series in the quaternionic case (Theorem 3.5).

## 2 Preliminaries

### 2.1 Notation and definitions

Let  $\mathbb{H}$  be Hamiltonian quaternions and  $\mathcal{O}$  the Hurwitz order (cf. [6]). The half-space of quaternions of degree  $n$  is defined as

$$H(n; \mathbb{H}) := \{ Z = X + iY \mid X, Y \in \text{Her}_n(\mathbb{H}), Y > 0 \}.$$

Let  $J_n := \begin{pmatrix} O_n & 1_n \\ -1_n & O_n \end{pmatrix}$ . Then, the group of symplectic similitudes

$$\{ M \in M(2n, \mathbb{H}) \mid {}^t \overline{M} J_n M = q J_n \text{ for some positive } q \in \mathbb{R} \}$$

acts on  $H(n; \mathbb{H})$  by

$$Z \mapsto M \langle Z \rangle = (AZ + B)(CZ + D)^{-1}, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Let  $\Gamma_n$  denote the modular group of quaternions of degree  $n$  defined by

$$\begin{aligned} G_n &:= \{ M \in M(2n, \mathbb{H}) \mid {}^t \overline{M} J_n M = J_n \}, \\ \Gamma_n &:= \Gamma_n(\mathcal{O}) = M(2n, \mathcal{O}) \cap G_n. \end{aligned}$$

For a given  $q \in \mathbb{N}$ , the congruence subgroup  $\Gamma_0^{(n)}(q)$  of  $\Gamma_n$  is defined by

$$\Gamma_0^{(n)}(q) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \mid C \equiv O_n \pmod{qM(n, \mathcal{O})} \right\}.$$

In this subsection,  $\Gamma$  always denotes either  $\Gamma_n$  or  $\Gamma_0^{(n)}(q)$ .

Let  $1 = e_1, e_2, e_3, e_4$  denote the canonical basis of  $\mathbb{H}$ , which is characterized by the identities

$$e_4 = e_2 e_3 = -e_3 e_2, \quad e_2^2 = e_3^2 = -1.$$

We consider the canonical isomorphism

$$M(n, \mathbb{H}) \longrightarrow M(2n, \mathbb{C})$$

given by  $\check{A} = (\check{a}_{ij})$ , where  $\check{a} = \begin{pmatrix} a_1 + a_2 i & a_3 + a_4 i \\ -a_3 + a_4 i & a_1 - a_2 i \end{pmatrix}$ , if  $a = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4$  (cf. [6]).

We use the above isomorphism to define  $\det(A)$  for  $A \in M(n, \mathbb{H})$ . For a similitude  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$  and a function  $f : H(n; \mathbb{H}) \longrightarrow \mathbb{C}$ , we define the slash operator  $|_k$  by

$$(f|_k M)(Z) = \det(M)^{\frac{k}{2}} \det(CZ + D)^{-k} f((AZ + B)(CZ + D)^{-1}).$$

A holomorphic function  $f : H(n; \mathbb{H}) \longrightarrow \mathbb{C}$  is called a *quaternionic modular form of degree  $n$  and weight  $k$  for  $\Gamma$*  if  $f$  satisfies

$$(f|_k M)(Z) = f(Z),$$

for all  $M \in \Gamma$ . (The cusp condition is required if  $n = 1$ .)

We denote by  $M_k(\Gamma)$  the  $\mathbb{C}$ -vector space of all quaternionic modular forms of degree  $n$  and weight  $k$  for  $\Gamma$ . A modular form  $f \in M_k(\Gamma)$  possesses a Fourier expansion of the form

$$f(Z) = \sum_{0 \leq H \in \text{Her}_n^\tau(\mathcal{O})} a_f(H) e^{2\pi i \tau(H, Z)}, \quad Z \in H(n; \mathbb{H}),$$

where  $\text{Her}_n^\tau(\mathcal{O})$  denotes the dual lattice of  $\text{Her}_n(\mathcal{O}) := \{S \in M(n, \mathcal{O}) \mid {}^t \overline{S} = S\}$  with respect to the reduced trace form  $\tau$  (cf. [6]). For simplicity, we put  $q^H := e^{2\pi i \tau(H, Z)}$  for  $H \in \text{Her}_n^\tau(\mathcal{O})$ . Using this notation, we write the above Fourier expansion simply as  $f = \sum_H a_f(H) q^H$ .

For an even integer  $k$ , we consider the Eisenstein series

$$E_k^{(n)}(Z) := \sum_{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{n0} \setminus \Gamma_n} \det(CZ + D)^{-k}, \quad Z \in H(n; \mathbb{H}), \quad (2.1)$$

where  $\Gamma_{n0} := \left\{ \begin{pmatrix} A & B \\ O_n & D \end{pmatrix} \in \Gamma_n \right\}$ . It is well known that this series belongs to  $M_k(\Gamma_n)$  if  $k > 4n - 2$ .

We call this series the *quaternionic Eisenstein series of degree  $n$  and weight  $k$* .

## 2.2 Fourier coefficients of Eisenstein series

In this section, we introduce an explicit formula for the Fourier coefficients of the degree 2 quaternionic Eisenstein series obtained by Krieg (cf. [7]).

Let  $k > 6$  be an even integer and let

$$E_k^{(2)}(Z) = \sum_{0 \leq H \in \text{Her}_2^\tau(\mathcal{O})} a_k(H) e^{2\pi i \tau(H, Z)}$$

be the Fourier expansion of the degree 2 quaternionic Eisenstein series  $E_k^{(2)}$ . According to [7], we introduce an explicit formula for  $a_k(H)$ . Given  $O_2 \neq H \in \text{Her}_2^\tau(\mathcal{O})$ , the “greatest common divisor” of  $H$  is given by

$$\varepsilon(H) := \max\{d \in \mathbb{N} \mid d^{-1}H \in \text{Her}_2^\tau(\mathcal{O})\}.$$

**Theorem 2.1** (Krieg [7]). *Let  $k > 6$  be even and  $H \neq O_2$ . Then, the Fourier coefficient  $a_k(H)$  is given by:*

$$a_k(H) = \sum_{0 < d \mid \varepsilon(H)} d^{k-1} \alpha^*(2\det(H)/d^2)$$

and

$$\alpha^*(\ell) = \begin{cases} -\frac{2k}{B_k} & \text{if } \ell = 0, \\ -\frac{4k(k-2)}{(2^{k-2}-1)B_k B_{k-2}} [\sigma_{k-3}(\ell) - 2^{k-2}\sigma_{k-3}(\ell/4)] & \text{if } \ell \in \mathbb{N}, \end{cases}$$

where  $B_m$  is the  $m$ -th Bernoulli number and

$$\sigma_k(m) := \begin{cases} 0 & \text{if } m \notin \mathbb{N}, \\ \sum_{0 < d \mid m} d^k & \text{if } m \in \mathbb{N}. \end{cases}$$

### 2.3 $U(p)$ -operator

In the remainder of this paper, we assume that  $p$  is an odd prime. For a formal power series of the form  $F = \sum_H a_F(H) q^H$ , we define a  $U(p)$  type operator as

$$U(p) : F = \sum_H a_F(H) q^H \mapsto F|U(p) := \sum_H a_F(pH) q^H.$$

In particular, for a modular form  $F \in M_k(\Gamma_0^{(n)}(p))$ , we may regard  $U(p)$  as a Hecke operator (cf. [2], [7]). We prove this in this section. More precisely, we prove that

**Proposition 2.2.** *If  $F \in M_k(\Gamma_0^{(n)}(p))$  then  $F|U(p) \in M_k(\Gamma_0^{(n)}(p))$ .*

To prove this proposition, we introduce the following lemma.

**Lemma 2.3.** *A complete set of representatives for the left cosets of*

$$\Gamma_0^{(n)}(p) \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix} \Gamma_0^{(n)}(p)$$

is given by

$$\left\{ \begin{pmatrix} O_n & -1_n \\ 1_n & T \end{pmatrix} \mid T \in \text{Her}_n(\mathcal{O})/p\text{Her}_n(\mathcal{O}) \right\}.$$

*Proof of Lemma 2.3.* We set  $\gamma_T := \begin{pmatrix} O_n & -1_n \\ 1_n & T \end{pmatrix}$  and prove

$$\Gamma_0^{(n)}(p) \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix} \Gamma_0^{(n)}(p) = \bigcup_{T \in \text{Her}_n(\mathcal{O})/p\text{Her}_n(\mathcal{O})} \Gamma_0^{(n)}(p) \gamma_T.$$

By decomposition

$$\begin{pmatrix} O_n & -1_n \\ 1_n & T \end{pmatrix} = \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix} \begin{pmatrix} 1_n & T \\ O_n & 1_n \end{pmatrix}, \quad (2.2)$$

we easily see the inclusion

$$\Gamma_0^{(n)}(p) \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix} \Gamma_0^{(n)}(p) \supset \bigcup_{T \in \text{Her}_n(\mathcal{O})/p\text{Her}_n(\mathcal{O})} \Gamma_0^{(n)}(p) \gamma_T. \quad (2.3)$$

We shall prove the converse inclusion. Note that  $T \equiv T' \pmod{p\text{Her}_n(\mathcal{O})}$  if and only if  $\Gamma_0^{(n)}(p) \gamma_T = \Gamma_0^{(n)}(p) \gamma_{T'}$ . Hence, we have

$$\bigcup_{T \in \text{Her}_n(\mathcal{O})/p\text{Her}_n(\mathcal{O})} \Gamma_0^{(n)}(p) \gamma_T = \bigcup_{T \in \text{Her}_n(\mathcal{O})} \Gamma_0^{(n)}(p) \gamma_T$$

as a set. Again, by the decomposition (2.2), it suffices to show that, for any  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(p)$ , there exists  $S \in \text{Her}_n(\mathcal{O})$  such that

$$\begin{pmatrix} O_n & -1_n \\ 1_n & T \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} O_n & -1_n \\ 1_n & S \end{pmatrix}^{-1} \in \Gamma_0^{(n)}(p).$$

A direct calculation shows that

$$\begin{pmatrix} O_n & -1_n \\ 1_n & T \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} O_n & -1_n \\ 1_n & S \end{pmatrix}^{-1} = \begin{pmatrix} -CS + D & -C \\ (A + TC)S - (B + TD) & A + TC \end{pmatrix}.$$

Hence, the proof is reduced to finding  $S \in \text{Her}_n(\mathcal{O})$  such that  $AS \equiv B + TD \pmod{pM(n, \mathcal{O})}$ . Recall that  $A^t \overline{D} - B^t \overline{C} = 1_n$  and hence  $A^t \overline{D} \equiv 1_n \pmod{pM(n, \mathcal{O})}$ . If we choose  $S$  as  $S := {}^t \overline{D}(B + TD)$ , then  $AS \equiv B + TD \pmod{pM(n, \mathcal{O})}$ . To complete the proof, we need to show that  $S = {}^t \overline{D}(B + TD) \in \text{Her}_n(\mathcal{O})$ . This assertion comes from the fact that  ${}^t \overline{D}B, {}^t \overline{D}TD \in \text{Her}_n(\mathcal{O})$ .  $\square$

We now return to the proof of Proposition 2.2.

*Proof of Proposition 2.2.* Let  $F \in M_k(\Gamma_0^{(n)}(p))$ . From Lemma 2.3, we have

$$\begin{aligned} F|_{\Gamma_0^{(n)}(p)} \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix} \Gamma_0^{(n)}(p) &= \sum_T F|_k \begin{pmatrix} O_n & -1_n \\ 1_n & T \end{pmatrix} \\ &= \sum_T F|_{W_p|_k} \begin{pmatrix} 1_n & T \\ O_n & p1_n \end{pmatrix}, \end{aligned}$$

where  $W_p$  is the Fricke involution

$$F \mapsto F|_{W_p} := F|_k \begin{pmatrix} O_n & -1_n \\ p1_n & O_n \end{pmatrix}.$$

We see by the usual way that  $F|_{W_p} \in M_k(\Gamma_0^{(n)}(p))$ . If we write  $G = F|_{W_p} = \sum_H a_G(H) q^H$ , then

$$\begin{aligned} \sum_T F|_{W_p|_k} \begin{pmatrix} 1_n & T \\ O_n & p1_n \end{pmatrix} &= \sum_T G|_k \begin{pmatrix} 1_n & T \\ O_n & p1_n \end{pmatrix} \\ &= \sum_H \left( \sum_T e^{\frac{2\pi i}{p} \tau(H, T)} \right) a_G(H) e^{\frac{2\pi i}{p} \tau(H, Z)} \\ &= c \cdot G|_U(p), \end{aligned}$$

where  $c := \sharp \text{Her}_n(\mathcal{O})/p\text{Her}_n(\mathcal{O})$  and the last equality follows from the following lemma.

**Lemma 2.4.** For fixed  $H \in \text{Her}_n^\tau(\mathcal{O})$ , we have

$$\sum_T e^{\frac{2\pi i}{p}\tau(H,T)} = \begin{cases} 0 & \text{if } H \notin p\text{Her}_n^\tau(\mathcal{O}), \\ c & \text{if } H \in p\text{Her}_n^\tau(\mathcal{O}). \end{cases} \quad (2.4)$$

*Proof of Lemma 2.4.* For  $H \in \text{Her}_n^\tau(\mathcal{O})$ , we define

$$G(H) := \sum_{T \in \text{Her}_n(\mathcal{O})/p\text{Her}_n(\mathcal{O})} e^{\frac{2\pi i}{p}\tau(H,T)}.$$

This definition is independent of the choice of the representation  $T$ . Replacing  $T$  by  $T + S$ , we obtain

$$G(H) = G(H) e^{\frac{2\pi i}{p}\tau(H,S)}.$$

Hence,  $G(H) = 0$  unless  $e^{\frac{2\pi i}{p}\tau(H,S)} = 1$ ; i.e.,  $\tau(H,S) \in p\mathbb{Z}$ . This implies  $\tau(\frac{1}{p}H, S) \in \mathbb{Z}$  for all  $S \in \text{Her}_n(\mathcal{O})$ . The definition of a dual lattice yields

$$\frac{1}{p}H \in \text{Her}_n^\tau(\mathcal{O}).$$

□

From this lemma, we have

$$F|I_0^{(n)}(p) \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix} I_0^{(n)}(p) = c \cdot F|W_p|U(p).$$

Hence, the action of  $U(p)$  is described by the action of the double coset

$$I_0^{(n)}(p) \begin{pmatrix} 1_n & O_n \\ O_n & p1_n \end{pmatrix} I_0^{(n)}(p).$$

Therefore, we have  $F|U(p) \in M_k(I_0^{(n)}(p))$ , which completes the proof of Proposition 2.2. □

**Remark 2.5.** The proof of Lemma 2.4 is due to Krieg.

### 3 Main results

#### 3.1 Modularity of $p$ -adic Eisenstein series

In this subsection, we deal with a suitable constant multiple of the normalized quaternionic Eisenstein series

$$G_k = G_k^{(2)} := (2^{k-2} - 1) \frac{B_k B_{k-2}}{4k(k-1)} E_k^{(2)}$$

and show that certain  $p$ -adic limits of this Eisenstein series are “real” modular forms for  $I_0^{(2)}(p)$ .

We write the Fourier expansion of  $G_k$  as  $G_k = \sum_H b_k(H) q^H$ . We remark that

$$b_k(O_2) = (2^{k-2} - 1) \frac{-B_k B_{k-2}}{4k(k-2)}.$$

For an odd prime  $p$  we put

$$\begin{aligned} G_k^* &:= \frac{-1}{1+p^{k-3}} \left\{ p^{2(k-3)} (G_k|U(p) - p^{k-1}G_k) - (G_k|U(p) - p^{k-1}G_k)|U(p) \right\} \\ &\in M_k(I_0^{(n)}(p)), \end{aligned}$$

where this modularity follows from Proposition 2.2. The first main theorem is

**Theorem 3.1.** *Let  $p$  be an odd prime and  $k$  an even integer with  $k \geq 4$ . Define a sequence  $\{k_m\}$  by*

$$k_m := k + (p-1)p^{m-1}.$$

*Then, the corresponding sequence of Eisenstein series  $\{G_{k_m}\}$  has a  $p$ -adic limit  $G_k^*$  and we have*

$$\lim_{m \rightarrow \infty} G_{k_m} = G_k^* \in M_k(\Gamma_0^{(2)}(p)). \quad (3.1)$$

*Proof.* The proof of (3.1) is reduced to show that  $G_k^*$  is obtained by removing all  $p$ -factors of the Fourier coefficients of the quaternionic Eisenstein series.

To calculate the Fourier coefficients of  $G_k^*$ , we set

$$F_k = G_k|U(p) - p^{k-1}G_k.$$

We can then rewrite  $G_k^*$  as

$$G_k^* = \frac{-1}{1+p^{k-3}}(p^{2(k-3)}F_k - F_k|U(p)).$$

We write the Fourier expansions as

$$G_k^* = \sum_H A_k(H)q^H, \quad F_k = \sum_H B_k(H)q^H.$$

First, we calculate the constant term of  $G_k^*$ . Since

$$b_k(O_2) = (2^{k-2} - 1) \frac{-B_k B_{k-2}}{4k(k-2)},$$

the constant term of  $G_k^*$  becomes

$$\begin{aligned} A_k(O_2) &= \frac{-1}{1+p^{k-3}} \{p^{2(k-3)}(b_k(O_2) - p^{k-1}b_k(O_2)) - (b_k(O_2) - p^{k-1}b_k(O_2))\} \\ &= (1-p^{k-1})(1-p^{k-3})(2^{k-2} - 1) \frac{-B_k B_{k-2}}{4k(k-2)}. \end{aligned}$$

Second, we calculate the coefficient  $A_k(H)$  for  $H$  with  $\text{rank}(H) = 1$ .

$$\begin{aligned} B_k(H) &= b_k(pH) - p^{k-1}b_k(H) \\ &= (2^{k-2} - 1) \frac{B_{k-2}}{2(k-2)} \left( \sum_{0 < d|p\varepsilon(H)} d^{k-1} - p^{k-1} \sum_{0 < d|\varepsilon(H)} d^{k-1} \right) \\ &= (2^{k-2} - 1) \frac{B_{k-2}}{2(k-2)} \sigma_{k-1}^*(\varepsilon(H)), \end{aligned}$$

where  $\sigma_m^*(N)$  is defined as

$$\sigma_m^*(N) := \sum_{\substack{0 < d|N \\ (p,d)=1}} d^m.$$

Note that  $B_k(pH) = B_k(H)$  when  $\text{rank}(H) = 1$ . Hence, we have

$$A_k(H) = (1-p^{k-3})(2^{k-2} - 1) \frac{B_{k-2}}{2(k-2)} \sigma_{k-1}^*(\varepsilon(H)).$$

Finally, we consider the case  $\text{rank}(H) = 2$ .

$$\begin{aligned}
B_k(H) &= b_k(pH) - p^{k-1}b_k(H) \\
&= \sum_{0 < d | p\varepsilon(H)} d^{k-1} [\sigma_{k-3} \left( \frac{2p^2 \det H}{d^2} \right) - 2^{k-2} \sigma_{k-3} \left( \frac{2p^2 \det H}{4d^2} \right)] \\
&\quad - p^{k-1} \sum_{0 < d | \varepsilon(H)} d^{k-1} [\sigma_{k-3} \left( \frac{2 \det H}{d^2} \right) - 2^{k-2} \sigma_{k-3} \left( \frac{2 \det H}{4d^2} \right)] \\
&= \sum_{\substack{0 < d | \varepsilon(H) \\ (p,d)=1}} d^{k-1} [\sigma_{k-3} \left( \frac{2p^2 \det H}{d^2} \right) - 2^{k-2} \sigma_{k-3} \left( \frac{2p^2 \det H}{4d^2} \right)].
\end{aligned}$$

Here, the last equality was obtained from the elemental property:

**Lemma 3.2.** *Let  $p$  be a prime and  $N$  a positive integer. For a function  $f : \mathbb{N} \rightarrow \mathbb{N}$ , the following holds:*

$$\sum_{0 < d | pN} f(d) = \sum_{\substack{0 < d | N \\ (p,d)=1}} f(d) + \sum_{0 < d | N} f(pd).$$

Therefore,

$$\begin{aligned}
A_k(H) &= \frac{-1}{1+p^{k-3}} (p^{2(k-3)} B_k(pH) - B_k(H)) \\
&= \frac{-1}{1+p^{k-3}} \left( p^{2(k-3)} \sum_{\substack{0 < d | \varepsilon(H) \\ (p,d)=1}} d^{k-1} [\sigma_{k-3} \left( \frac{2p^2 \det H}{d^2} \right) - 2^{k-2} \sigma_{k-3} \left( \frac{2p^2 \det H}{4d^2} \right)] \right. \\
&\quad \left. - \sum_{\substack{0 < d | \varepsilon(H) \\ (p,d)=1}} d^{k-1} [\sigma_{k-3} \left( \frac{2p^4 \det H}{d^2} \right) - 2^{k-2} \sigma_{k-3} \left( \frac{2p^4 \det H}{4d^2} \right)] \right).
\end{aligned}$$

By repeatedly applying Lemma 3.2, we obtain

$$p^{2m} \sigma_m(N) - \sigma_m(p^2 N) = -(1+p^m) \sum_{\substack{0 < d | N \\ (p,d)=1}} d^m.$$

From this, we have

$$A_k(H) = \sum_{\substack{0 < d | \varepsilon(H) \\ (p,d)=1}} d^{k-1} [\sigma_{k-3}^* \left( \frac{2 \det H}{d^2} \right) - 2^{k-2} \sigma_{k-3}^* \left( \frac{2 \det H}{4d^2} \right)].$$

Summarizing these calculations, we obtain the following formula:

**Proposition 3.3.** *The following holds:*

$$A_k(H) = \begin{cases} (1-p^{k-1})(1-p^{k-3})(2^{k-2}-1) \frac{-B_k B_{k-2}}{4k(k-2)}, & \text{if } H = O_2, \\ (1-p^{k-3})(2^{k-2}-1) \frac{B_{k-2}}{2(k-2)} \sigma_{k-1}^*(\varepsilon(H)), & \text{if } \text{rank}(H) = 1, \\ \sum_{\substack{0 < d | \varepsilon(H) \\ (p,d)=1}} d^{k-1} [\sigma_{k-3}^* \left( \frac{2 \det H}{d^2} \right) - 2^{k-2} \sigma_{k-3}^* \left( \frac{2 \det H}{4d^2} \right)], & \text{if } \text{rank}(H) = 2. \end{cases}$$

On the other hand,

$$b_{k_m}(H) = \begin{cases} (2^{k_m-2} - 1) \frac{-B_{k_m} B_{k_m-2}}{4k_m(k_m-2)}, & \text{if } H = O_2, \\ (2^{k_m-2} - 1) \frac{B_{k_m-2}}{2(k_m-2)} \sigma_{k_m-1}(\varepsilon(H)), & \text{if } \text{rank}(H) = 1, \\ \sum_{0 < d | \varepsilon(H)} d^{k_m-1} [\sigma_{k_m-3} \left( \frac{2 \det H}{d^2} \right) - 2^{k_m-2} \sigma_{k_m-3} \left( \frac{2 \det H}{4d^2} \right)], & \text{if } \text{rank}(H) = 2. \end{cases}$$

Combining these formulas and the Kummer congruence, we can prove that

$$\lim_{m \rightarrow \infty} b_{k_m}(H) = A_k(H)$$

for all  $H \in \text{Her}_2^{\tau}(\mathcal{O})$ . This completes the proof of Theorem 3.1.  $\square$

**Remark 3.4.** Following Hida [3], our  $G_k^*$  can be  $p$ -adic analytically interpolated with respect to the weight.

### 3.2 Transcendental $p$ -adic Eisenstein series

As we have seen in the previous section, under certain conditions, a  $p$ -adic limit of a quaternionic Eisenstein series becomes a “real” modular form with rational Fourier coefficients. This also holds for Siegel Eisenstein and Hermitian Eisenstein series. More precisely, they coincide with the genus theta series (cf. [4], [11]). In these cases (Siegel, Hermitian cases), the  $p$ -adic Eisenstein series is algebraic. We shall show that there exists an example of a transcendental  $p$ -adic Eisenstein series for quaternionic modular forms.

The second main theorem is

**Theorem 3.5.** Let  $p$  be an odd prime and  $\{k_m\}$  the sequence defined by

$$k_m := 2 + (p-1)p^{m-1}.$$

Then, the  $p$ -adic Eisenstein series  $\tilde{E} = \lim_{m \rightarrow \infty} E_{k_m}^{(2)}$  is transcendental; namely,  $\tilde{E}$  has transcendental coefficients where  $E_k^{(2)}$  is the normalized quaternionic Eisenstein series of degree 2 defined in (2.1).

*Proof.* We calculate  $\tilde{a}(H) := \lim_{m \rightarrow \infty} a_{k_m}(H)$  at  $H = \begin{pmatrix} 1 & \frac{e_1+e_2}{2} \\ \frac{e_1-e_2}{2} & 1 \end{pmatrix} \in \text{Her}_2^{\tau}(\mathcal{O})$ . The convergence for general  $H$  is proved similarly.

It follows from Theorem 2.1 that

$$a_{k_m}(H) = -\frac{4k_m(k_m-2)}{(2^{k_m-2} - 1)B_{k_m}B_{k_m-2}}.$$

(We note that  $\varepsilon(H) = 1$  and  $\det(H) = \frac{1}{2}$ .) We rewrite the right-hand side as

$$-4 \cdot \frac{2 + (p-1)p^{m-1}}{B_{2+(p-1)p^{m-1}}} \cdot \frac{1}{B_{(p-1)p^{m-1}}} \cdot \frac{p^m}{2^{(p-1)p^{m-1}} - 1} \cdot \frac{p-1}{p}$$

and calculate the  $p$ -adic limit separately:

$$(i) \quad \lim_{m \rightarrow \infty} \frac{2 + (p-1)p^{m-1}}{B_{2+(p-1)p^{m-1}}} = \frac{B_2}{2} = \frac{1}{12}.$$

This is a consequence of the  $p$ -adic Kummer congruence.

$$(ii) \quad \lim_{m \rightarrow \infty} B_{(p-1)p^{m-1}} = \frac{p-1}{p}.$$



This identity comes from the fact that the residue of the  $p$ -adic  $L$ -function  $L_p(s, \chi^0)$  at  $s = 0$  is just  $1 - \frac{1}{p}$ .

$$(iii) \quad \lim_{m \rightarrow \infty} \frac{2^{(p-1)p^{m-1}} - 1}{p^m} = \frac{\log_p(2^{p-1})}{p}.$$

where  $\log_p$  is the  $p$ -adic logarithmic function defined by

$$\log_p(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots, \quad (|x|_p < 1).$$

Leopoldt's formula [9] states that

$$\lim_{m \rightarrow \infty} \frac{x^{p^m} - 1}{p^m} = \log_p(x).$$

if  $|x - 1|_p < 1$ . This implies that

$$\lim_{m \rightarrow \infty} \frac{2^{(p-1)p^{m-1}} - 1}{p^m} = \frac{1}{p} \cdot \log_p(2^{p-1}).$$

Combining these formulas, we obtain

$$\tilde{a}(H) = \lim_{m \rightarrow \infty} a_{k_m}(H) = \frac{-48p}{\log_p(2^{p-1})}. \quad (3.2)$$

We shall show that  $\log_p(2^{p-1})$  is transcendental. Let  $\exp_p$  be the  $p$ -adic exponential function defined by

$$\exp_p(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad (|x|_p < p^{-\frac{1}{p-1}}).$$

It is known that if  $|x|_p < p^{-\frac{1}{p-1}}$ , then

$$\exp_p(\log_p(1 + x)) = 1 + x, \quad (\text{e.g., [8]}).$$

To prove the transcendency of  $\log_p(2^{p-1})$ , we use the following theorem by Mahler:

**Theorem 3.6** (Mahler [10]). *Let  $\mathbb{C}_p$  be the completion of the algebraic closure of  $\mathbb{Q}_p$ . For any algebraic over  $\mathbb{Q}$   $p$ -adic number  $\alpha \in \mathbb{C}_p$  with  $0 < |\alpha|_p < p^{-\frac{1}{p-1}}$ , the quantity  $\exp_p(\alpha)$  is transcendental.*

We note that  $|x|_p < p^{-\frac{1}{p-1}}$  is equivalent to  $|x|_p < 1$  for odd prime  $p$  (e.g., [8], p.114). We put  $\alpha = 2^{p-1} - 1$ . Since  $|\alpha|_p < 1$ , we have

$$\exp_p(\log_p(1 + \alpha)) = 1 + \alpha = 2^{p-1}.$$

The right-hand side is obviously algebraic. Hence, by Mahler's theorem,  $\log_p(1 + \alpha) = \log_p(2^{p-1})$  must be transcendental. Thus, we can prove the transcendency of  $\tilde{a}(H)$  at  $H = \begin{pmatrix} 1 & \frac{e_1 + e_2}{2} \\ \frac{e_1 - e_2}{2} & 1 \end{pmatrix}$ . This completes the proof of Theorem 3.5.  $\square$

**Remark 3.7.** By the above proof, we see that all coefficients  $\tilde{a}(H)$  corresponding to  $H$  with rank 2 are transcendental. However,  $\tilde{a}(H)$  for  $H$  with  $\text{rank}(H) \leq 1$  are rational.

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Toshiyuki KIKUTA  
 Department of Mathematics  
 Interdisciplinary Graduate School of  
 Science and Engineering Kinki University  
 Higashi-Osaka 577-8502, Japan  
 E-mail: kikuta84@gmail.com

Shoyu NAGAOKA  
 Department of Mathematics  
 School of Science and Engineering  
 Kinki University  
 Higashi-Osaka 577-8502, Japan  
 E-mail: nagaoka@math.kindai.ac.jp